Name ____

Teacher (circle): Kelly Kresser Rahman

Class block (circle): A B C D G

Honors Advanced Math
Final Exam 2004 - Solutions

Lexington High School
Mathematics Department

This is a 90-minute exam, but you will be allowed to work for up to 120 minutes.

The exam has 3 parts. Directions for each part appear below.

In total, there are 66 points that you can earn. A letter grade scale will be set by the course faculty after the tests have been graded.

Part A. Short Problems

7 questions, 2 points each, 14 points total

You must write your answers in the answer boxes.

If your answer is correct, you will receive full credit. Showing work is not required.

If your answer is incorrect, you may receive half credit if you have shown some correct work.

A good pace on this part would be to spend around 3 minutes per problem.

Part B. Medium Problems

5 problems, 4 points each, 20 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend 5-6 minutes per problem.

Part C. Long Problems

4 problems, 8 points each, 32 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend around 10 minutes per problem.

Part A. Short Problems

7 problems, 2 points each, 14 points total

1. Construct a sinusoidal function $f(x)$ that rises from a minimum point at (4, 6) to a maximum point at (7, 11).

Solution: amplitude = 2.5; period = 6, wave axis = 8.5, so:

involving cosine: $2.5\cos(\pi/3(x-7))+8.5$

involving sine: $2.5\sin(\pi/3(x-5.5))+8.5$

Answer to question 1:

$f(x)$ = either of the functions above or algebraic equivalent.

2. For what values of k will the second-degree equation $2y^2+kxy+x^2+x-5=0$ be a hyperbola?

Solution: will be a hyperbola when $B^2-4AC>0$, so when $k^2-4(1)(2)>0$ or when $k<-\sqrt{8}$ or $k>\sqrt{8}$.

Answer to question 2:

The interval for k indicated above, using any notation or description.

3. A sequence t_1, t_2, t_3, … is defined by this explicit formula: $t_n=n^2$, where $n \geq 1$.
Write a **recursive definition** for the same sequence.

Solution: the sequence looks like *1, 4, 9, 16, …*, so each successive term is found by adding $2n–1$ to the previous term, or $t_n=t_{n-1}+2n-1$ with $t_1=1$.

Answer to question 3:

A good recursive definition should contain both a recursive formula and the first term.

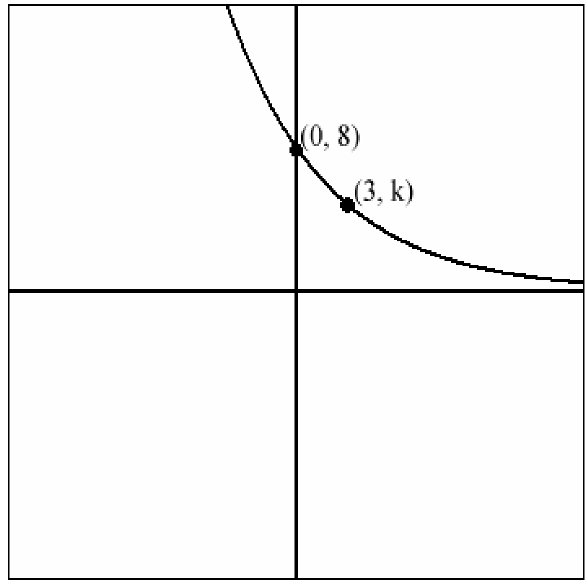
4. The matrix equation below represents a composition of transformations that maps point (x,y) to an image point (x', y'). What are the three transformations that are composed?

$$[x\ y]\cdot\begin{bmatrix}\cos 32 & -\sin 32\\ \sin 32 & \cos 32\end{bmatrix}\cdot\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\cdot\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=[x'\ y']$$

Answer to question 4:

The three transformations are **rotation by 32 degrees clockwise** followed by **rotation by 90 degrees clockwise** followed by **reflection over the line $y = x$**.

5. Write a function formula for the exponential function $F(x)$ whose graph is shown.



Solution: The model should be $F(x) = ab^x$ giving $F(0) = a(1) = 8$ and $F(3) = ab^3 = k$. The solution to the system yields:

$F(x) = 8(k/8)^{x/3}$.

Answer to question 5:

$F(x)$ = the above function or equivalent.

6. Determine the quotient and remainder when $x^4 + x^2 - 5x + 1$ is divided by $x^2 - 5x + 1$.

Solution:

Just set up the long division (a helpful addition is to add the $0x^3$ term), like this:

$$x^2 - 5x + 1 \overline{\big)\, x^4 + 0x^3 + x^2 - 5x + 1}$$

This results in a quotient of $x^2 + 5x + 25$ and a remainder of $115x - 24$.

Answer to question 6:

The quotient is _____$x^2 + 5x + 25$______and the remainder is _____$115x - 24$_______.

7. Find the solutions of the equation $x^3 - i = 0$, expressed as complex numbers in polar form.

Solution: This is equivalent to solving the equation $x^3 = i$, which, in turn, is equivalent to finding the 3 cube roots of i. This is done most easily in complex polar form, where $i = 1cis90^\circ$. By DeMoivre's Theorem:

$(1\text{cis}90^\circ)^{1/3} = 1^{1/3}\text{cis}(\frac{90^\circ + 360^\circ k}{3})$, for k = 0, 1, 2. This yields:

$(1\text{cis}90^\circ)^{1/3} = 1\text{cis}30^\circ, 1\text{cis}150^\circ$, and $1\text{cis}270^\circ$.

Answer to question 7:

All three of the complex numbers above are cube roots of i. Notice that one of them is the purely complex number, $-i$.

Part B. Medium Problems

5 problems, 4 points each, 20 points total

8. Given vectors **v** = <3, 4> and **w** = <5, –12>, compute the following.

a. Find the angle between **v** and **w**, rounded to the nearest 0.01 radian.

Solution: Use the formula (definition of dot product) $\cos\theta = \frac{v \cdot w}{|v||w|}$. The angle is approximately 2.10 radians.

b. Find a *unit vector* (vector of magnitude 1) **u** that is perpendicular to **w**.

Solution: We can set up a couple equations to solve for the components of u.
Define **u** = <*x*, *y*>. Then, from the magnitude of 1: $\sqrt{x^2 + y^2} = 1$ and from the perpendicular fact: $< x, y > \cdot < 5,-12 >= 0$ which implies $5x - 12y = 0$. Solving this system yields: $< x, y >=< \frac{12}{13}, \frac{5}{13} >$ or, equivalently, $< x, y >=< -\frac{12}{13}, -\frac{5}{13} >$.

9. Consider the functions: $f(x) = \dfrac{x^2 - k^2}{x - k}$ and $g(x) = \log_k x$, where $k > 1$.

Determine whether each of the following statements is **True** or **False**. Circle **True** or **False** within each box.

~~**True**~~ or **False** $\lim_{x \to k} f(x)$ does not exist.	~~**True**~~ or **False** $\lim_{x \to k^+} f(x) = f(k)$.	~~**True**~~ or **False** $\lim_{x \to -\infty} f(x) = \infty$.	~~**True**~~ or **False** *f*(*k*) = 2*k*.
~~**True**~~ or **False** *f*(*x*) is continuous at $x = k$.	**True** or ~~**False**~~ $\lim_{x \to 0^+} g(x) = -\infty$.	**True** or ~~**False**~~ $\lim_{x \to \infty} g(x) = \infty$.	~~**True**~~ or **False** $g(k^3) = 3k$.

10. Given the polynomial $P(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$.

a. According to the Rational Zeros (Roots) Theorem, what are the possible rational zeros of *P*(*x*) ?

Solution: By the RZT, $\left\{\dfrac{\text{factors of } 4}{\text{factors of } 2}\right\} = \left\{\pm \tfrac{1}{2}, 1, 2, 4\right\}$.

b. Find all complex zeros of *P*(*x*) using analytic or symbolic methods.
Hint to get you started: Note that *P*(–2*i*) = 0.

Solution: *P*(–2*i*) = 0 implies that –2*i* is zero. Since *p* has real coefficients, the complex conjugates theorem indicates that 2*i* is also a zero. Two methods are possible to find the remaining two complex zeros: *i.* use 2*i* and –2*i* to find a quadratic term and use long division to find the remaining quadratic that includes the other two zeroes. *ii.* Plug in the rational zeros from part **a** using the remainder theorem until you find two rational zeroes. Either way, the other two zeros are 1 and –1/2.

11. Bart and Lisa play a game where they take turns rolling a standard 6-sided die. The first person to roll a 4 wins.

a. If Bart rolls first, what is his probability of winning the game?

Solution: It is best to draw a tree diagram until you see the pattern. Bart's probability of winning is given by the convergent infinite geometric series: $\frac{1}{6}+\frac{1}{6}\cdot(\frac{5}{6})^2+\frac{1}{6}\cdot(\frac{5}{6})^4+\cdots$ The sum of this series is 6/11.

b. Suppose Bart and Lisa play this game 100 times with Bart always rolling first. For each game both of them put \$1 into a jackpot and the first person to roll a 4 wins the jackpot. If they both start with \$100, based on expected values how much money should each of them have after the 100 games have been played.

Solution: Use expected value to find the expected gain/loss for Lisa from each game. The probability of Lisa winning is 5/11, so her expected value is:

EV(LISA) = (6/11)(–1)+(5/11)(+1) = –1/11 of a dollar per game. If Lisa plays 100 games, she will lose $100(1/11) \approx$ \$9.10, leaving her with \$90.90. Bart will necessarily win what Lisa loses, so Bart will have \$109.10.

12. Find all solutions to the equation $4\sin 2x = -3$ in the interval $0 \le x \le 2\pi$. You may not use a graphical method, but the use of the inverse function (with results rounded to the nearest .01 radian) is acceptable.

Solution: $4\sin 2x = -3 \Rightarrow \sin 2x = -3/4 \Rightarrow 2x = \sin^{-1}(-3/4)$

$$\Rightarrow 2x = \{5.44 \text{ or } 3.99\} + 2\pi k\text{ , } k = 0,1.$$

$$\Rightarrow x = \{2.72 \text{ or } 2.00\} + \pi k\text{ , } k = 0,1.$$

The four solutions in the given interval are: $\{2.00,\ 2.72,\ 5.14,\ 5.86\}$.

Part C. Long Problems

4 problems, 8 points each, 32 points total

13. For this problem assume your friend has loaned you a graphing calculator (TI-83 or other similar calculator) that has certain broken features.

a. The **ln** key on the calculator is broken. How could you evaluate $\ln 4$ using only the remaining features of the calculator?

Solution:

Using the change-of-base formula: ln 4 = (log 4)/(log e) ≈ 1.386.

OR

Since ln 4 is the solution to the equation $e^x = 4$, solve this equation graphically to get x = ln 4 ≈ 1.386.

b. The calculator's ***nPr*** function does not work. How could you evaluate ${}_{15}P_6$ using a method that includes the use of the ***nCr*** function of the calculator?

Solution: ${}_{15}P_6 = {}_{15}C_6 \cdot 6! = 5005 \cdot 120 = 3603600$.

c. The **cos** key on the calculator is broken. How could you evaluate $\cos(\frac{\pi}{7})$ using only the remaining features of the calculator?

Solution:

$\cos(\pi/7) = \sin(\pi/7)/\tan(\pi/7) \approx 0.901$

OR

$\cos(\pi/7) = \sqrt{1-\sin^2(\pi/7)} \approx 0.901$.

OR

$\cos(\pi/7) = \sin(\pi/2 - \pi/7) = \sin(5\pi/14) \approx 0.901$.

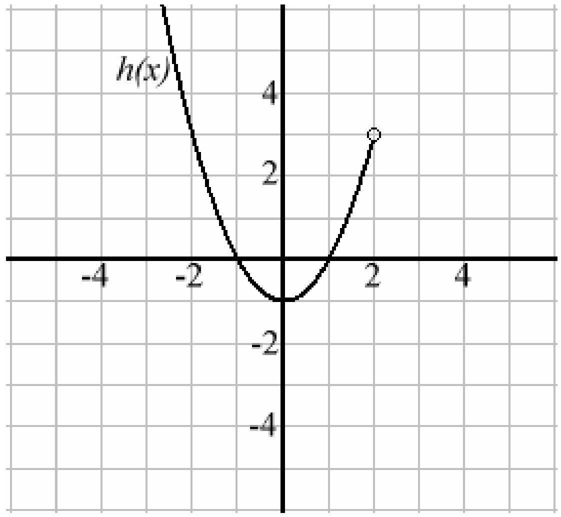
d. Suppose you need to know the value of e accurate to five decimal places, and you don't remember it. The broken features from parts **a-c** have been fixed, but now the **e** and **e^x** operations on the calculator are broken. How could you still get the desired approximation of e from the calculator?

Solution:

$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$, so scroll downward in the table of function $\left(1+\frac{1}{x}\right)^x$ until you see that the function's values converge to 2.71828.

14. Consider functions $g(x)$, $h(x)$, and $j(x)$ as defined by the following equation, graph and table.

$g(x) = 1 + \sqrt{x+2}, x \geq -2$



x	$j(x)$
–2	4
–1	2
0	0
1	–2
2	–4

a. Find these values:

$(g \circ j \circ h)(-1) =$ **Solution:** $g(j(h(-1))) = g(j(0)) = g(0) = 1 + \sqrt{2}$

$(g \circ j)(2) =$ **Solution:** $g(j(2)) = g(-4)$ which is undefined

b. Find $g^{-1}(x)$ and its domain and range. If $g^{-1}(x)$ does not exist, explain why.

Solution: $g^{-1}(x) = (x-1)^2 - 2$, with domain $x \geq 1$ and range $g^{-1}(x) \geq -2$.

c. Let $f(x) = h(x) - g(x)$. For what values of x is $f(x)$ positive?

Solution: $f(x)$ is positive where $h(x) > g(x)$, which is true only for $-2 \leq x \leq -1.618$.

d. On the graph of $h(x)$ above, sketch the figure that represents a reflection of $h(x)$ across the line $y = x$. Is the figure the graph of $h^{-1}(x)$? Explain why or why not.

Solution: No. The reflected figure is not a function at all, because it fails the vertical line test. To get an inverse function $h^{-1}(x)$ it is necessary to use a restriction of $h(x)$ that is one-to-one, such as a restriction to $x \leq 0$.

15. A park has the shape of a quadrilateral. On a coordinate grid measured in feet, the vertices of the park are (400, 100), (0, 50), (0, 200), and (200, 250).

a. There is a walking path along each diagonal of the park. Find the lengths of these two paths.

Solution:

Length of diagonal d_1 from (0, 50) to (200, 250) is $\sqrt{200^2 + 200^2} \approx 282.843$ feet.

Length of diagonal d_2 from (0, 200) to (400, 100) is $\sqrt{400^2 + 100^2} \approx 412.311$ feet.

b. The two paths along the diagonals intersect at a point inside the park. Find the measure of one of the angles formed at the intersection of the paths.

Solution:

Diagonal d_1 has slope 1, so an angle of inclination of $\tan^{-1}(1) = 45°$.

Diagonal d_2 has slope –1/4, so an angle of inclination of $\tan^{-1}(-1/4) = -14.036°$.

Therefore one angle between the diagonals is 45° + 14.036° = 59.036°.
[In radians: 0.785 + 0.245 = 1.030 radians. Other angle is 120.964° or 2.111 radians.]

(There are other methods, such as finding various angles of triangles, or using a formula for the angle between two lines.)

c. Find the area of the park.

Solution:

The easiest method is to take the area of the rectangle $0 \le x \le 400$ and $50 \le y \le 250$, and subtract the areas of three triangle regions, leaving the area of the quadrilateral.
Quadrilateral area = 80000 – 5000 – 15000 – 10000 = 50000.

(There are other methods, such as subdividing the quadrilateral into triangles and finding the areas of the triangles.)

16. Consider a circle, **C**, centered at (0, 0) with parametric equations $\begin{cases} x_C(t) = 3\cos(t) \\ y_C(t) = 3\sin(t) \end{cases}$.

Also consider an ellipse, **E**, centered at (2, –1) with a horizontal major axis of length 9 and a vertical minor axis of length 6.

a. Write Cartesian (rectangular) equations for **C** and **E**.

equation for **C**

Solution:

$x^2 + y^2 = 9$

equation for **E**

Solution:

$$\left(\frac{x-2}{4.5}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$$

b. Write parametric equations for **E** of the form $\begin{cases} x_E(t) = \cdots \\ y_E(t) = \cdots \end{cases}$

Solution:

$$\begin{cases} x_E(t) = 4.5\cos(t) + 2 \\ y_E(t) = 3\sin(t) - 1 \end{cases}$$

c. Write a pair of equations expressing $x_E(t)$ in terms of $x_C(t)$ and $y_E(t)$ in terms of $y_C(t)$.

Solution:

$x_E(t) = 1.5\, x_C(t) + 2$

$y_E(t) = y_C(t) - 1$

d. Write a single matrix equation that relates the coordinate matrices $[x_C(t) \quad y_C(t)]$ and $[x_E(t) \quad y_E(t)]$.

Solution:

$$[x_E(t) \quad y_E(t)] = [x_C(t) \quad y_C(t)] \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} + [2 \quad -1].$$