Math League SCASD
Study Packets

Meet #1

Number Theory

Self-study Packet

2016 -2017

Problem Categories for this Meet:

1. Mystery: Problem solving
2. Geometry: Angle measures in plane figures including supplements and complements
3. **Number Theory: Divisibility rules, factors, primes, composites**
4. Arithmetic: Order of operations; mean, median, mode; rounding; statistics
5. Algebra: Simplifying and evaluating expressions; solving equations with 1 unknown including identities

Information you need to know about NUMBER THEORY...

Divisibility Rules

A number is divisible by:

- 2 if its ones digit is even (0, 2, 4, 6, 8)
- 3 if the sum of its digits is divisible by 3 (for example, to check if 364 is divisible by 3, add 3 + 6 + 4. You get 13. 13 is not divisible by 3, so 364 is not either).
- 4 if the number formed by its last two digits is divisible by 4 (for example, to check if 2,320 is divisible by 4, look at the number formed by the last two digits; in this case, 20. 20 is divisible by 4, so 2,320 is as well).
- 5 if its ones digit is 0 or 5
- 6 if it divisible by 2 *and* 3
- 8 if the number formed by its last *three* digits is divisible by 8 (similar to the rule for 4)
- 9 if the sum of its digits is divisible by 9 (similar to the rule for 3)
- 10 if its ones digit is 0

Factoring

To find the factors of a number, it is useful to find the *pairs* of numbers that multiply to give you that number. Using the divisibility tricks, it is much easier to narrow this down. I recommend starting with 1. You can stop when you pass the square root of the number, because you have found all the factors.

For example, list all the factors of 120.

1 x 120	2 x 60	3 x 40	4 x 30	5 x 24
6 x 20	8 x 15	10 x 12		

The factors of 120 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120.

Primes and Composites

Prime Number: any number with **exactly** two factors

Composite Number: any number with **more than** two factors

** 0 and 1 are neither prime nor composite!

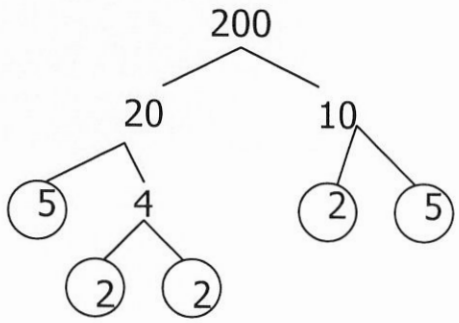
It would be VERY useful to memorize all the prime numbers under 100 (Questions where you need to know them occur over and over again all season). They are:

2	3	5	7	11	13	17
19	23	29	31	37	41	43
47	53	59	61	67	71	73
79	83	89	97			

<u>Prime Factoring</u>: Two common ways to find the prime factorization of a number are to use a factor tree or to use the ladder method. An example of each follows:

Factor Tree: Find the prime factorization of 200.

1. Start with your original number.
2. Find a pair of numbers that multiply to give you your original number.
3. Find a pair of numbers that multiply to give you each of the factors.
4. Continue until you only have prime numbers.
5. The prime factorization is written as a product of all the primes.



$200 = 2^3 \times 5^2$

Ladder Method: Find the prime factorization of 60.

1. Start with the lowest prime number (2) and check if your number is divisible by it.
2. If it is, divide your number by that. If not, try 3, then 5, then 7, etc, until you find a number that is a factor of your original number. Divide by it.
3. Repeat by dividing your new number by its smallest prime factor.
4. Continue until you are left with 1.
5. The prime factorization is written as a product of all its primes, which are conveniently ordered on the left of your ladder.

2	60
2	30
3	15
5	5
	1

$60 = 2^2 \bullet 3 \bullet 5$

Category 3
Number Theory
Meet #1 - October, 2016

1) The factors of a positive whole number, N, include the numbers 1, N, and all positive numbers, W, such that N divided by W is a positive whole number. How many factors of 36 are also multiples of 4 ?

2) There are three whole numbers, A, B, and C, that are between 120 and 130 that are each the product of exactly two different prime numbers. What is the sum A + B + C ?

3)  **is a positive integer greater than 1.**

 **is the square of an integer.**

 **is the cube of an integer.**

What is the smallest possible number of different positive factors (divisors) that  **can have ?**

Answers

1) _____

2) _____

3) _____

Solutions to Category 3
Number Theory
Meet #1 - October, 2016

Answers

1) 3

2) 374

3) 7

1) The factors of 36 are: 1, 2, 3, 4, 6, 9, 12, 18, 36. Those factors that are multiples of four are: 4, 12, and 36. So, there are three.

2) 121 = 11 x 11 No - the factors must be different.
122 = 2 x 61 Yes - both 2 and 61 are prime.
123 = 3 x 41 Yes - both 3 and 41 are prime.
124 = 2 x 2 x 31 No - too many factors.
125 = 5 x 5 x 5 No - too many factors.
126 = 2 x 3 x 3 x 7 No - too many factors.
127 = prime No - prime
128 = 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 No - too many factors.
129 = 3 x 43 Yes - both 3 and 43 are prime.
Therefore, A + B + C = 122 + 123 + 129 = 374.

3) Sixty-four is the smallest positive integer that meets all the criteria. The factors of 64 are: 1, 2, 4, 8, 16, 32, and 64, so there are seven in all.

Category 3
Number Theory
Meet #1 - October, 2014

1) What is the sum of all the prime numbers between 40 and 60 ?

2)  **has an odd number of factors.**

 **is divisible by 3.**

 **> 20.**

 **< 70.**

What is the value of  **?**

3) How many of the three-digit numbers that can be made using the digits 1, 3, and 5 are not multiples of three ? Each digit can be used more than once.

Answers

1) _____

2) _____

3) _____

Solutions to Category 3
Number Theory
Meet #1 - October, 2014

1) **41 + 43 + 47 + 53 + 59 = 243**

Answers

1) 243

2) 36

3) 18

2)  **is a perfect square if it has an odd number of factors. If that square is between 20 and 70, then it could be any of these: 25, 36, 49, or 64. The only one divisible by 3 is 36.**

3) The full list of three-digit numbers is: 111, 113, 115, 131, 133, 135, 151, 153, 155, 311, 313, 315, 331, 333, 335, 351, 353, 355, 511, 513, 515, 531, 533, 535, 551, 553, and 555. Checking that there should be twenty-seven numbers: 3x3x3 = 27.

The numbers that are divisible by 3 are those whose digit-sum is a multiple of 3: 111, 135, 153, 315, 333, 351, 513, 531, and 555.

That is nine numbers. Therefore, there are 27 - 9, or 18 numbers, that are not multiples of three.

Category 3
Number Theory
Meet #1, October 2012

1. Two primes p and q have a sum of 38. Given that $p > q$, find the value of $p - q$.

2. What single-digit value of N will make the 7-digit number 1,295,N84 divisible by 18?

3. What number is the least three-digit multiple of 5 that has exactly six factors?

Answers

1. _____

2. _____

3. _____

Solutions to Category 3
Number Theory
Meet #1, October 2012

Answers
1. **24**
2. **7**
3. **175**

1. The two primes must be 31 and 7. Their positive difference is 31 – 7 = **24**.

2. The 7-digit number 1,295,N84 is clearly even, so we only need to make sure it is divisible by 9. The sum of the known digits is 1 + 2 + 9 + 5 + 8 + 4 = 29. N will have to be **7** to get us to the next multiple of 9.

3. Numbers with exactly six factors must have a prime factorization of the form p^5 or $p^2 \times q$, where p and q are primes. The smallest multiple of 5 of the form p^5 is $5^5 = 3125$, which is not three digits. The other options are the forms $p^2 \times 5$ or $5^2 \times q$. The least three-digit number of the $p^2 \times 5$ form is $7^2 \times 5 = 49 \times 5 = 245$. The least three-digit number of the $5^2 \times q$ form is $5^2 \times 7 = 25 \times 7 =$ **175**, which is our desired answer.

Category 3 – Number Theory

1. Thinking of the natural numbers from 2 to (and including) 20, what is the positive difference between the sum of all composites and the sum of all primes?

2. What value of the digit A will make the number $567{,}88A$ be divisible by 12?

3. Find the sum of all factors of the number 96 (including 1 and 96).

Answers

1. _____

2. _____

3. _____

Solutions to Category 3 – Number Theory

Answers	
1.	55
2.	8
3.	252

1. Sum of primes: $2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 = 77$

   Sum of composites:

   $4 + 6 + 8 + 9 + 10 + 12 + 14 + 15 + 16 + 18 + 20 = 132$

   The difference is $132 - 77 = 55$

2. In order to be divisible by 12, a number should be divisible both by 3 and by 4. The sum of digits of $567{,}88A$ is $(7 + A)$, so A can be $2, 5,$ or 8 in order to make it divisible by 3. However, only a value of 8 will make it divisible by 4.

3. The factors of 96 are:

   $96 = 1 \times 96 = 2 \times 48 = 3 \times 32 = 4 \times 24 = 6 \times 16 = 8 \times 12$

   The sum therefore is:

   $1 + 2 + 3 + 4 + 6 + 8 + 12 + 16 + 24 + 32 + 48 + 96 = 252$

   Of all numbers under 100, the number 96 has the largest sum of factors. As preparation for meet #2 try to figure out why.

Category 3
Number Theory
Meet #1, October 2008

1. The number 210 is divisible by how many prime numbers?

2. The seven digit number 2A5A756 is divisible by 3. What is the sum of the possible values of A?

3. The prime numbers 23 and 29 are the smallest two primes that are 6 apart and have no primes between them. What is the sum of the smallest two primes that are 8 apart and have no other primes between them?

Answers

1. _____
2. _____
3. _____

Solutions to Category 3
Number Theory
Meet #1, October 2008

Answers

1. 4

2. 12

3. 186

1. 210 is divisible by:
1, 2, 3, 5, 6, 7, 10, 14, 15, 21,20, 35, 42, 70, 105, and 210.
Of those only 2, 3, 5, and 7 are prime numbers, so there are **4.**

2. In order for 2A5A756 to be divisible by 3, the sum of the digits must be divisible by 3. The sum of the digits is 2A + 25 and that needs to be a multiple of 3. The smallest possible value of 2A + 25 is 2(0)+25 = 25 and the largest possible value is 2(9) + 25 = 43. The multiples of 3 between 25 and 43 are 27, 30, 33, 36, 39, and 42. We can solve the equation 2A + 25 = 27 or 30 or 33 or 36 or 39 or 42 to find A. Subtracting 25 gives us 2A = 2, 5, 8, 11, 14, or 17. We then divide by 2, but ignore any decimals since A is a digit. The possible values of A then are: 1, 4, or 7 and they have a sum of **12.**

3. Listing the primes we get :

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97……

Those last two are the first two that are 8 apart with no other primes between them. The sum is 89 + 97 = **186.**

Category 3
Number Theory
Meet #1, October 2006

1. What is the least three-digit number that is divisible by exactly three different prime numbers?

2. The five-digit number $35N2N$ is divisible by 2, 3, 4, 6, 8, 9, and 12, among other numbers. What is the value of the digit N?

3. How much greater is the product of the first four primes than the product of the first three composites?

Answers

1. _____
2. _____
3. _____

Solutions to Category 3
Number Theory
Meet #1, October 2006 *Average score: 1.5 answers correct*

Answers

1. 102

2. 4

3. 18

1. Some people might be fooled by $3 \times 5 \times 7 = 105$, but $2 \times 3 \times 17 =$ **102** is the least three-digit number that is divisible by exactly three primes.

2. N clearly must be an even digit, so 0, 2, 4, 6, and 8 are possible. For $35N2N$ to be divisible by 3, the sum of the digits must be a multiple of three. So far we have $3 + 5 + N + 2 + N = 10 + 2N$. Let's use the possible values of N to evaluate this expression. If $N = 0$, then $10 + 2N = 10 + 0 = 10$, which is not divisible by 3. If $N = 2$, then $10 + 2N = 10 + 4 = 14$, which is not divisible by 3. If $N = 4$, then $10 + 2N = 10 + 8 = 18$, which *is* divisible by 3. Neither $N = 6$ nor $N = 8$ gives a multiple of 3, so 4 is our only candidate. Let's make sure that the number 35424 is divisible by all the other numbers listed. It is divisible by 4 since the last two digits 24 are divisible by 4. It is divisible by 6 since it passes the test for 2 and 3. It is divisible by 8 since the last three digits 424 are divisible by 8. It is divisible by 9 since the sum of the digits (18) is divisible by 9. Finally, it is divisible by 12 since it passes the test for 3 and 4. The digit N is indeed **4**.

Some Incorrect Answers Seen

1. 105, 110, 120, 1001

2. 9

3. 186, 6

3. The product of the first four primes is $2 \times 3 \times 5 \times 7 = 210$. The product of the first three composites is $4 \times 6 \times 8 = 192$. The difference is $210 - 192 =$ **18**.